Chapter 6. Central Force Motion

(Most of the material presented in this chapter is taken from Thornton and Marion, Chap. 8, and Goldstein, Poole, and Safko Chap. 3)

In this chapter we will study the problem of two bodies moving under the influence of a mutual central force.

6.1 Reduction to the Equivalent One-body Problem – the Reduced Mass

We consider a system consisting of two point masses, m_1 and m_2, when the only forces are those due to an interaction potential U. We will assume that U is a function of the distance between the two particles $r = |\mathbf{r}| = |\mathbf{r}_2 - \mathbf{r}_1|$. Such a system has six degrees of freedom. We could choose, for example, the three components of each of the two vectors, $\mathbf{r}_1$ and $\mathbf{r}_2$ (see Figure 6-1). However, since the potential energy is solely a function of the distance between the two particles, i.e., $U = U(r)$, it is to our advantage if we also express the kinetic energy as function of $\mathbf{r}$ (that is, of $\dot{\mathbf{r}}$). Let's first define the center of mass $\mathbf{R}$ of the system as

$$(m_1 + m_2)\mathbf{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2. \tag{6.1}$$

We also consider the distance between each particle and the center of mass

$$\begin{aligned}
\mathbf{r}_1 - \mathbf{R} &= \mathbf{r}_1 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = -\frac{m_2}{m_1 + m_2}\mathbf{r} \\
\mathbf{r}_2 - \mathbf{R} &= \mathbf{r}_2 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m_1}{m_1 + m_2}\mathbf{r},
\end{aligned} \tag{6.2}$$

with $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.

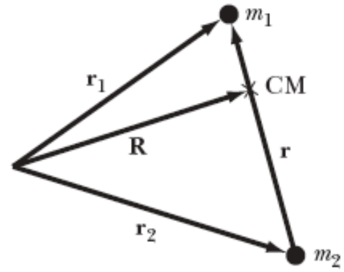

Figure 6-1 – The different vectors involved in the two-body problem.

We now calculate the kinetic energy of the system

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{\mathbf{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\mathbf{r}}_2^{\,2} \\ &= \frac{1}{2}m_1\left[\dot{\mathbf{R}} + \left(\dot{\mathbf{r}}_1 - \dot{\mathbf{R}}\right)\right]^2 + \frac{1}{2}m_2\left[\dot{\mathbf{R}} + \left(\dot{\mathbf{r}}_2 - \dot{\mathbf{R}}\right)\right]^2, \end{aligned} \tag{6.3}$$

and inserting equations (6.2) in equation (6.3) we get

$$\begin{aligned} T &= \frac{1}{2}m_1\left[\dot{\mathbf{R}} - \frac{m_2}{m_1+m_2}\dot{\mathbf{r}}\right]^2 + \frac{1}{2}m_2\left[\dot{\mathbf{R}} + \frac{m_1}{m_1+m_2}\dot{\mathbf{r}}\right]^2 \\ &\frac{1}{2}m_1\left[\dot{\mathbf{R}}^2 - 2\frac{m_2}{m_1+m_2}\dot{\mathbf{r}}\bullet\dot{\mathbf{R}} + \left(\frac{m_2}{m_1+m_2}\right)^2\dot{\mathbf{r}}^2\right] \\ &+\frac{1}{2}m_2\left[\dot{\mathbf{R}}^2 + 2\frac{m_1}{m_1+m_2}\dot{\mathbf{r}}\bullet\dot{\mathbf{R}} + \left(\frac{m_1}{m_1+m_2}\right)^2\dot{\mathbf{r}}^2\right] \\ &= \frac{1}{2}\left(m_1+m_2\right)\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\mathbf{r}}^2. \end{aligned} \tag{6.4}$$

We introduce a new quantity μ, the **reduced mass**, defined as

$$\mu = \frac{m_1 m_2}{m_1+m_2}, \tag{6.5}$$

which can alternatively be written as

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{6.6}$$

We can use equation (6.5) to write the kinetic energy as

$$T = \frac{1}{2}\left(m_1+m_2\right)\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2. \tag{6.7}$$

We have, therefore, succeeded in expressing the kinetic energy as a function of $\dot{\mathbf{r}}$. We are now in a position to write down the Lagrangian for the central force problem

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}\left(m_1+m_2\right)\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U\left(r\right), \end{aligned} \tag{6.8}$$

where the potential energy $U(r)$ is yet undefined, except for the fact that it is solely a function of the distance between the two particles. It is, however, seen from equation (6.8) that the three components of the center of mass vector are cyclic. More precisely,

$$\frac{\partial L}{\partial X} = \frac{\partial L}{\partial Y} = \frac{\partial L}{\partial Z} = 0, \tag{6.9}$$

with $\mathbf{R} = X\mathbf{e}_x + Y\mathbf{e}_y + Z\mathbf{e}_z$. The center of mass is, therefore, either at rest or moving uniformly since the equations of motion for $X, Y,$ and Z can be combined into the following vector relation

$$(m_1 + m_2)\ddot{\mathbf{R}} = 0. \tag{6.10}$$

Since the center of mass vector (and its derivative) does not appear anywhere else in the Lagrangian, we can drop the first term of the right hand side of equation (6.8) in all subsequent analysis and only consider the remaining three degrees of freedom. The new Lagrangian is therefore

$$L = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r). \tag{6.11}$$

What is left of the Lagrangian is exactly what would be expected if we were dealing with a problem of a single particle of mass μ subjected to a fixed central force. Thus, the central force motion of two particles about their common center of mass is reducible to an equivalent one-body problem.

6.2 The First Integrals of Motion

Since we are dealing with a problem where the force involved is conservative, where the potential is a function of the distance r of the reduced mass to the force center alone, the system has spherical symmetry. From our discussion of Noether's theorem in Chapter 4 (cf., section 4.7.3) we know that for such systems the angular momentum is conserved (cf., section 4.7.5). That is,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = cste. \tag{6.12}$$

From the definition of the angular momentum itself we know that $\mathbf{L}$ is always parallel to vectors normal to the plane containing $\mathbf{r}$ and $\mathbf{p}$. Furthermore, since in this case $\mathbf{L}$ is fixed, it follows that the motion is at all time confined to the aforementioned plane. We are, therefore, fully justified to use polar coordinates as the two remaining generalized coordinates for this problem (i.e., we can set the third generalized coordinate, say, z to be a constant since the motion is restricted to a plane). Since $\mathbf{r} = r\mathbf{e}_r$, we have

$$
\begin{aligned}
\dot{\mathbf{r}} &= \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r \\
&= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta.
\end{aligned} \tag{6.13}
$$

In the last equation we have used the following relation (with the usual transformation between the (r,θ) polar and the (x,y) Cartesian coordinates)

$$
\begin{aligned}
\mathbf{e}_r &= \cos(\theta)\mathbf{e}_x + \sin(\theta)\mathbf{e}_y \\
\mathbf{e}_\theta &= -\sin(\theta)\mathbf{e}_x + \cos(\theta)\mathbf{e}_y,
\end{aligned} \tag{6.14}
$$

from which it can be verified that

$$
\dot{\mathbf{e}}_r = \frac{\partial \mathbf{e}_r}{\partial r}\dot{r} + \frac{\partial \mathbf{e}_r}{\partial \theta}\dot{\theta} = \dot{\theta}\mathbf{e}_\theta. \tag{6.15}
$$

We can now rewrite the Lagrangian as a function of r and θ

$$
L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r). \tag{6.16}
$$

We notice from equation (6.16) that θ is a cyclic variable. The corresponding generalized momentum is therefore conserved, that is

$$
p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} = cste. \tag{6.17}
$$

The momentum p_θ is a **first integral of motion** and is seen to equal the magnitude of the angular momentum vector. It is customarily written as

$$
\boxed{l \equiv \mu r^2\dot{\theta} = cste} \tag{6.18}
$$

As the particle (i.e., the reduced mass) moves along its trajectory through an infinitesimal angular displacement $d\theta$ within an amount of time dt, the area dA swept out by its radius vector $\mathbf{r}$ is given by

$$
dA = \frac{1}{2}r^2 d\theta = \frac{1}{2}r^2\dot{\theta}\,dt. \tag{6.19}
$$

Alternatively, we can define the **areal velocity** as

$$
\begin{aligned}
\frac{dA}{dt} &= \frac{1}{2} r^2 \dot{\theta} \\
&= \frac{l}{2\mu} = cste.
\end{aligned}
\tag{6.20}
$$

Thus, the areal velocity is constant in time. This result, discovered by Kepler for planetary motion, is called **Kepler's Second Law**. It is important to realize that the conservation of the areal velocity is a general property of central force motion and is not restricted to the inverse-square law force involved in planetary motion.

Another first integral of motion (the only one remaining) concerns the conservation of energy. The conservation is insured because we are considering conservative systems. Writing E for the energy we have

$$
\begin{aligned}
E &= T + U \\
&= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r),
\end{aligned}
\tag{6.21}
$$

or using equation (6.18)

$$
\boxed{E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)}
\tag{6.22}
$$

6.3 The Equations of Motion

We will use two different ways for the derivation of the equations of motion. The first one consists of inverting equation (6.22) and express $\dot{r}$ as a function of E, l, and $U(r)$ such that

$$
\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left[E - U(r)\right] - \frac{l^2}{\mu^2 r^2}},
\tag{6.23}
$$

or alternatively

$$
dt = \pm\frac{dr}{\sqrt{\frac{2}{\mu}\left[E - U(r)\right] - \frac{l^2}{\mu^2 r^2}}}.
\tag{6.24}
$$

Equation (6.24) can be solved, once the potential energy $U(r)$ is defined, to yield the solution $t = t(r)$, or after inversion $r = r(t)$. We are, however, also interested in determining the shape of the path (or orbit) taken by the particle. That is, we would like to evaluate $r = r(\theta)$ or $\theta = \theta(r)$. To do so we use the following relation

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr. \tag{6.25}$$

Inserting equations (6.18) and (6.23) into equation (6.25), we get

$$\theta(r) = \pm\int \frac{\left(l/r^2\right)dr}{\sqrt{2\mu\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)}} + cste. \tag{6.26}$$

It is important to note that the integral given by equation (6.26) can be solved analytically only for certain forms of potential energy. Most importantly, if the potential energy is of the form $U(r) \propto r^n$, for $n = 2, -1,$ and -2 the solution is expressible in terms of circular functions.

The second method considered here for solving the equations of motion uses the Lagrange equations

$$\begin{aligned} \frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= 0 \\ \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= 0. \end{aligned} \tag{6.27}$$

The second of these equations was already used to get equation (6.18) for the conservation of angular momentum. Applying the first of equations (6.27) to the Lagrangian (equation (6.16)) gives

$$\mu\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\partial U}{\partial r} \equiv F(r). \tag{6.28}$$

We now modify this equation by making the following change of variable

$$u \equiv \frac{1}{r}. \tag{6.29}$$

We calculate the first two derivatives of u relative to θ

$$\begin{aligned} \frac{du}{d\theta} &= -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} \\ &= -\frac{\dot{r}}{r^2}\left(\frac{l}{\mu r^2}\right)^{-1} = -\frac{\mu}{l}\dot{r}, \end{aligned} \tag{6.30}$$

where we have used the fact that $\dot{\theta} = l/\mu r^2$, and for the second derivative

$$\begin{aligned}\frac{d^2u}{d\theta^2} &= \frac{d}{d\theta}\left(-\frac{\mu}{l}\dot{r}\right) = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{\mu}{l}\dot{r}\right) = -\frac{\mu}{l}\frac{\ddot{r}}{\dot{\theta}} \\ &= -\frac{\mu^2}{l^2}r^2\ddot{r}.\end{aligned} \tag{6.31}$$

From this equation we have

$$\ddot{r} = -\frac{l^2}{\mu^2}u^2\frac{d^2u}{d\theta^2}, \tag{6.32}$$

and from equation (6.18)

$$r\dot{\theta}^2 = r\left(\frac{l}{\mu r^2}\right)^2 = \frac{l^2}{\mu^2}u^3. \tag{6.33}$$

Inserting equations (6.32) and (6.33) in equation (6.28) yields

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F\left(\frac{1}{u}\right), \tag{6.34}$$

which can be rewritten as

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r)} \tag{6.35}$$

Equation (6.35) can be used to find the force law that corresponds to a known orbit $r = r(\theta)$.

Examples

1. Let's consider the case where the orbit is circular, i.e., $r = cste$. Then from equation (6.35) we find that

$$\frac{\mu r^2}{l^2}F(r) = cste. \tag{6.36}$$

Equation (6.36) implies that

$$F(r) \propto \frac{1}{r^2}, \tag{6.37}$$

which is the functional form of the gravitational force.

2. Let's assume that we have an orbit given by

$$r = ke^{\alpha\theta}. \tag{6.38}$$

From the second derivative of equation (6.38) relative to θ we have

$$\begin{aligned} \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) &= \frac{d^2}{d\theta^2}\left(\frac{e^{-\alpha\theta}}{k}\right) = -\frac{\alpha}{k}\frac{d}{d\theta}\left(e^{-\alpha\theta}\right) \\ &= \alpha^2 \frac{e^{-\alpha\theta}}{k} = \frac{\alpha^2}{r} \end{aligned} \tag{6.39}$$

and using equation (6.35), we get

$$F(r) = -\frac{l^2}{\mu r^2}\left(\frac{\alpha^2}{r} + \frac{1}{r}\right) \propto \frac{1}{r^3}. \tag{6.40}$$

6.4 Centrifugal Energy and the Effective Potential

In equations (6.23) to (6.26) for dr, dt, and θ, respectively, appeared a common term containing different energies (i.e., the total, potential, and rotational energies)

$$\sqrt{E - U(r) - \frac{l^2}{2\mu r^2}}. \tag{6.41}$$

The last term is the energy of rotation since

$$\frac{l^2}{2\mu r^2} = \frac{1}{2}\mu r^2 \dot{\theta}^2. \tag{6.42}$$

It is interesting to note that if we arbitrarily define this quantity as a type of "potential energy" U_c, we can derive a conservative force from it. That is, if we set

$$U_c \equiv \frac{l^2}{2\mu r^2}, \tag{6.43}$$

then the force associated with it is

$$F_c = -\frac{\partial U_c}{\partial r} = \frac{l^2}{\mu r^3} = \mu r\dot{\theta}^2. \tag{6.44}$$

The force defined by equation (6.44) is the so-called **centrifugal force**. It would, therefore, be probably better to call U_c is the **centrifugal potential energy** and to include it with the potential energy $U(r)$ to form the **effective potential energy** $V(r)$ defined as

$$\boxed{V(r) \equiv U(r) + \frac{l^2}{2\mu r^2}} \tag{6.45}$$

If we take for example the case of an inverse-square-law (e.g., gravity or electrostatic), we have

$$F(r) = -\frac{k}{r^2}, \tag{6.46}$$

and

$$U(r) = -\int F(r)\,dr = -\frac{k}{r}. \tag{6.47}$$

The effective potential is

$$V(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2}. \tag{6.48}$$

It is to be noted that the centrifugal potential "reduces" the effect of the inverse-square-law on the particle. This is because the inverse-square-law force is attractive while the centrifugal force is repulsive. This can be seen in Figure 6-2.

It is also possible to guess some characteristics of potential orbits simply by comparing the total energy with the effective potential energy at different values of r. From Figure 6-3 we can see that the motion of the particle is unbounded if the total energy (E_1) is greater than the effective potential energy for $r \geq r_1$ when $E_1 = V(r_1)$. This is because the positions for which $r < r_1$ are not allowed since the value under the square root in equation (6.41) becomes negative, which from equation (6.23) would imply an imaginary velocity. For the same reason, the orbit will be bounded with $r_2 \leq r \leq r_4$ for a total energy E_2, where $V(r_2) \leq E_2 \leq V(r_4)$. The *turning points* r_2 and r_4 are called the **apsidal distances**. Finally, the orbit is circular with $r = r_3$ when the total energy E_3 is such that $E_3 = V(r_3)$.

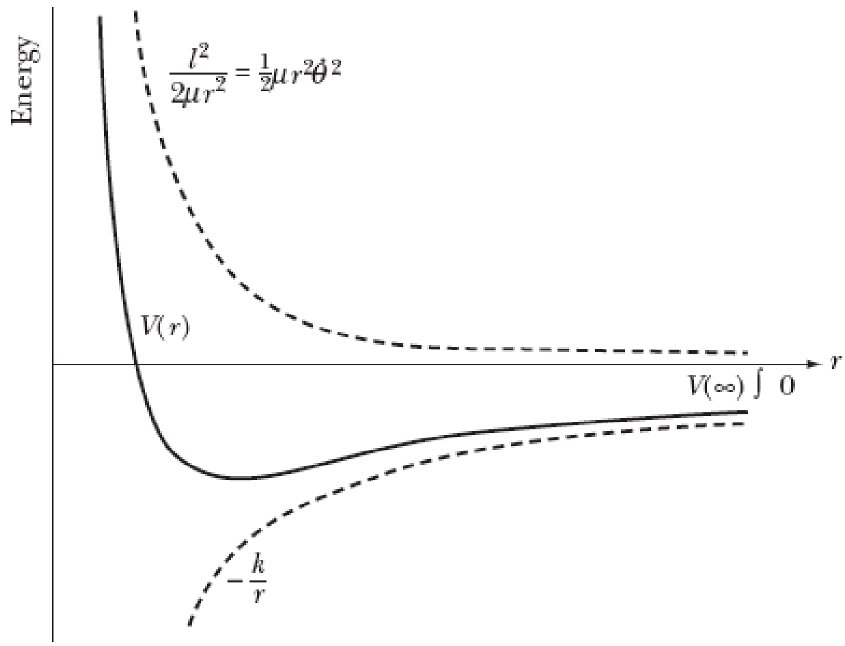

Figure 6-2 - Curves for the centrifugal, effective, and gravitational potential energies.

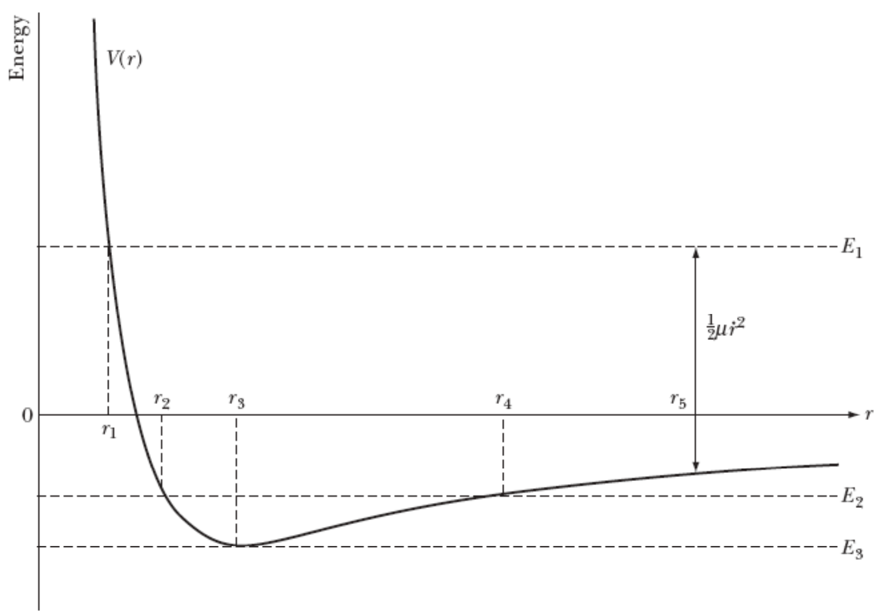

Figure 6-3 – Depending on the total energy, different orbits are found.

6.5 Planetary Motion – Kepler's Problem

The equation for a planetary orbit can be calculated from equation (6.26) when the functional form for the gravitational potential is substituted for $U(r)$

$$\theta(r) = \pm\int \frac{\left(l/r^2\right)dr}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\right)}} + cste. \tag{6.49}$$

This equation can be integrated if we first make the change of variable $u = 1/r$, the integral then becomes

$$\begin{aligned}\theta(u) &= \pm\int \frac{l\,du}{\sqrt{2\mu\left(E + ku - \frac{l^2u^2}{2\mu}\right)}} + cste \\ &= \pm\int \frac{du}{\sqrt{\frac{2\mu}{l^2}(E+ku) - u^2}} + cste,\end{aligned} \tag{6.50}$$

the sign in front of the integral has not changed as it is assumed that the limits of the integral were inverted when going from equation (6.49) to (6.50). We further transform equation (6.50) by manipulating the denominator

$$\begin{aligned}\theta(u) &= \pm\int \frac{du}{\sqrt{\frac{\mu^2k^2}{l^4} + \frac{2\mu E}{l^2} - \left(u - \frac{\mu k}{l^2}\right)^2}} + cste \\ &= \pm\int \frac{du}{\frac{\mu k}{l^2}\sqrt{\left(1 + \frac{2El^2}{\mu k^2}\right) - \left(\frac{ul^2}{\mu k} - 1\right)^2}} + cste \\ &= \pm\frac{\alpha}{\varepsilon}\int \frac{du}{\sqrt{1 - \left(\frac{\alpha u - 1}{\varepsilon}\right)^2}} + cste,\end{aligned} \tag{6.51}$$

with

$$\alpha \equiv \frac{l^2}{\mu k} \quad \text{and} \quad \varepsilon \equiv \sqrt{1 + \frac{2El^2}{\mu k^2}}. \tag{6.52}$$

Now, to find the solution to the integral of equation (6.51), let's consider a function $f(u)$ such that

$$f(u) = \cos\left[\theta(u)\right]. \tag{6.53}$$

Taking a derivative relative to u we get

$$\frac{df}{du} = -\sin(\theta)\frac{d\theta}{du}, \tag{6.54}$$

or

$$\begin{aligned}\frac{d\theta}{du} &= -\frac{1}{\sin(\theta)}\frac{df}{du} \\ &= -\frac{1}{\sqrt{1-\cos^2(\theta)}}\frac{df}{du} \\ &= -\frac{1}{\sqrt{1-f^2}}\frac{df}{du}.\end{aligned} \tag{6.55}$$

Returning to equation (6.51), and identifying $f(u)$ with the following

$$f(u) = \frac{\alpha u - 1}{\varepsilon}, \tag{6.56}$$

we find that

$$\mp\theta(u) = \cos^{-1}\left(\frac{\alpha u - 1}{\varepsilon}\right) + \beta, \tag{6.57}$$

or, since $\cos(\mp\theta) = \cos(\theta)$,

$$\frac{\alpha}{r} = 1 + \varepsilon\cos(\theta + \beta), \tag{6.58}$$

where β is some constant. If we choose r to be minimum when $\theta = 0$, then $\beta = 0$ and we finally have

$$\boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos(\theta)} \tag{6.59}$$

with

$$\alpha \equiv \frac{l^2}{\mu k} \quad \text{and} \quad \varepsilon \equiv \sqrt{1 + \frac{2El^2}{\mu k^2}}. \tag{6.60}$$

Equation (6.59) is that of a conic section with one focus at the origin. The quantities ε and 2α are called the **eccentricity** and the **latus rectum** of the orbit, respectively. The minimum value of r (when $\theta = 0$) is called the **pericenter**, and the maximum value for the radius is the **apocenter**. The turning points are **apsides**. The corresponding terms for motion about the sun are **perihelion** and **aphelion**, and for motion about the earth, **perigee** and **apogee**.

As was stated when discussing the results shown in Figure 6-3, the energy of the orbit will determine its shape. For example, we found that the radius of the orbit is constant when $E = V_{\min}$. We see from, equation (6.60) that this also implies that the eccentricity is zero (i.e., $\varepsilon = 0$). In fact, the value of the eccentricity is used to classify the orbits according to different conic sections (see Figure 6-4):

$\varepsilon > 1$	$E > 0$	Hyperbola
$\varepsilon = 1$	$E = 0$	Parabola
$0 < \varepsilon < 1$	$V_{\min} < E < 0$	Ellipse
$\varepsilon = 0$	$E = V_{\min}$	Circle

For planetary motion, we can determine the length of the major and minor axes (designated by $2a$ and $2b$) using equations (6.59) and (6.60). For the major axis we have

$$\begin{aligned} 2a = r_{\min} + r_{\max} &= \frac{\alpha}{1+\varepsilon} + \frac{\alpha}{1-\varepsilon} \\ &= \frac{2\alpha}{1-\varepsilon^2} = \frac{k}{|E|}. \end{aligned} \tag{6.61}$$

For the minor axis, we start by defining r_b and θ_b as the radius and angle where

$$b = r_b \sin(\theta_b). \tag{6.62}$$

Accordingly we have

$$r_{\min} - a = r_b \cos(\theta_b), \tag{6.63}$$

which can be written

$$a - r_{\min} = \frac{1}{2}\left[\frac{\alpha}{1-\varepsilon} - \frac{\alpha}{1+\varepsilon}\right] = \frac{\alpha\varepsilon}{1-\varepsilon^2}. \tag{6.64}$$

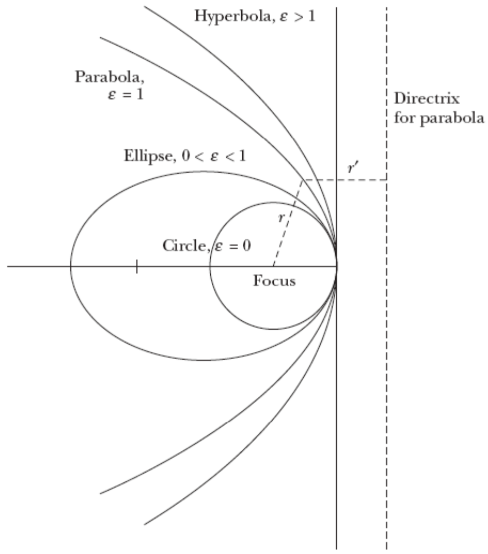

Figure 6-4 – The shape of orbits as a function of the eccentricity.

But from equation (6.59) we also have

$$r_b \cos(\theta_b) = \frac{\alpha \cos(\theta_b)}{1 + \varepsilon \cos(\theta_b)} = -\frac{\alpha\varepsilon}{1 - \varepsilon^2}, \tag{6.65}$$

or

$$\cos(\theta_b) = -\varepsilon, \tag{6.66}$$

and

$$r_b = \frac{\alpha}{1 - \varepsilon^2}. \tag{6.67}$$

Inserting equations (6.66) and (6.67) into equation (6.62) we finally get

$$b = \frac{\alpha}{\sqrt{1 - \varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}}, \tag{6.68}$$

or

$$b = \sqrt{\alpha a}. \tag{6.69}$$

The period of the orbit can be evaluated using equation (6.20) for the areal velocity

$$dt = \frac{2\mu}{l} dA. \tag{6.70}$$

Since the entire area enclosed by the ellipse will be swept during the duration of a period τ, we have

$$\int_0^{\tau} dt = \frac{2\mu}{l}\int_0^{A} dA, \tag{6.71}$$

or

$$\tau = \frac{2\mu}{l}A = \frac{2\mu}{l}\pi ab, \tag{6.72}$$

where we have the fact that the area of an ellipse is given by πab. Now, substituting the first of equations (6.60) and equation (6.69) in equation (6.72) we get

$$\boxed{\tau^2 = \frac{4\pi^2\mu}{k}a^3} \tag{6.73}$$

In the case of the motion of a solar system planet about the Sun we have

$$k = Gm_p m_s, \tag{6.74}$$

where G, m_p, and m_s are the universal gravitational constant, the mass of the planet, and the mass of the Sun, respectively. We therefore get

$$\begin{aligned}\tau^2 &= \frac{4\pi^2}{Gm_p m_s}\frac{m_p m_s}{m_p + m_s}a^3 \\ &= \frac{4\pi^2}{G\left(m_p + m_s\right)}a^3 \\ &\simeq \frac{4\pi^2 a^3}{Gm_s},\end{aligned} \tag{6.75}$$

since $m_s \gg m_p$. We find that the square of the period is proportional to the semi-major axis to the third, with the same proportionality constant for every planet. This (approximate) result is known as **Kepler's Third Law**. We end by summarizing Kepler's Laws

I. *Planets move in elliptical orbits about the Sun with the Sun at one focus.*
II. *The area per unit time swept out by a radius vector from the Sun to a planet is constant.*
III. *The square of a planet's period is proportional to the cube of the major axis of the planet's orbit.*